Holomorphic Dynamics - Lecture 13

Thm A If f maps $\mathcal{U}$, a Fatou component, to itself, then either $\mathcal{U}$ is:

1. attracting basin
2. parabolic basin
3. Siegel disk
4. Herman ring

ThmB (Classification of Fatou components)
Let $f:\hat{\mathbb{C}} \circlearrowleft$ be a rational map of degree $d \geq 2$. Then every connected component of the Fatou set is either:

1. attracting basin
2. parabolic basin
3. Siegel disk
4. Herman ring

Pf.: Let $\mathcal{U}$ a Fatou component.
If $\exists p$ s.t. $f^p(\mathcal{U}) = \mathcal{U}$, then apply Thm A to $f^p : \mathcal{U} \to \mathcal{U}$.

Otherwise: $f^i(\mathcal{U}) \cap f^j(\mathcal{U}) = \emptyset$ for all $i \neq j$. Then $\mathcal{U}$ is called a WANDERING COMPONENT.

Thm (Sullivan, no wandering domain)
A rational map f has no wandering components.

Rmk.: wandering domains DO exist for trascendental maps
e.g. $f(z) = e^z$

Idea of proof If $\mathcal{U}$ is a wandering domain, then you can perturb f along an "infinite-dimensional" space of directions.

 Teich $(\mathbb{D})$ is infinite dimensional

Since $\mathcal{U}$ is wandering, you are free to perturb the complex structure even if it's compatible with the dynamics f.
However: the space of <u>all</u> rational maps of given degree is finite-dimensional (idea: you

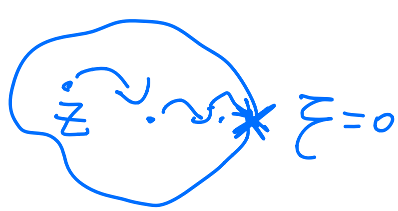
need to specify finitely many coefficients for $p(x)$ and finitely many for $q(x)$, if $f(x) = \frac{p(x)}{q(x)}$).

Proof of Thm A $f : \mathcal{U} \to \mathcal{U}$. Then either,

1. $\mathcal{U}$ contains an attracting fixed pt ATTRACT BASIN
2. every orbit $f^n(z) \to \xi \in \partial \mathcal{U}$ converges to the boundary
3. ~~f has finite order~~
4. f is conjugate to an irrational rotation of either a disk, a punctured disk or an annulus.

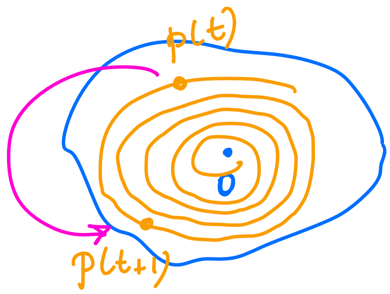
   SIEGEL DISK / HERMAN RING

(3) cannot happen : $f^m|_{\mathcal{U}} = id|_{\mathcal{U}}$ because f has only countably many periodic points

(2) 

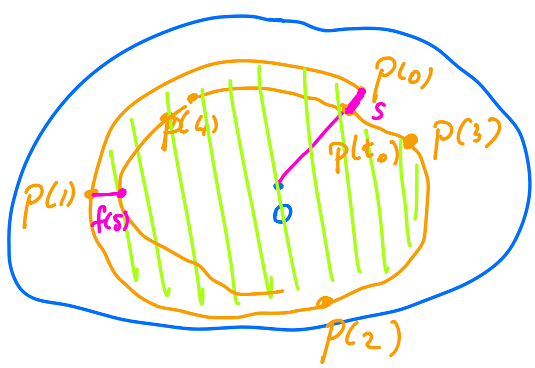
We need to prove that ξ is a parabolic fixed point.

Snail Lemma Let $p:(0,\infty)\to \mathcal{U}\setminus\{0\}$ be a path which converges to the origin, and suppose that $f(p(t)) = p(t+1)$. Then the origin is a fixed point of multiplier λ, with $|\lambda|<1$ or $\lambda = 1$.



Pf Since $p(n) = f^n(p(0))$ converges to 0, then 0 is NOT repelling. So $|\lambda|\leq 1$.

If $\lambda\neq 1$, then infinitesimally near 0, $f(z)\approx \lambda z$, hence $f^n(z)\approx \lambda^n z$ follows a spiral.



V = region delimited by $p([0,t_0])$ and transversal

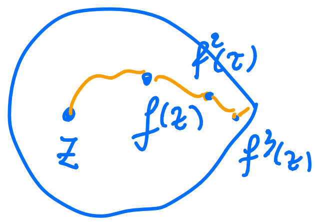
Then $f(V) \subsetneq V$

$$\begin{tikzcd} (V,0) \arrow[r, "f"] \arrow[d, phantom, "\cong"] & (V,0) \arrow[d, phantom, "\cong"] \\ (\mathbb{D},0) & (\mathbb{D},0) \end{tikzcd}$$

by Schwarz lemma, $|f'(0)| < 1$.

∎

If $f: \mathcal{U} \to \mathcal{U}$ and all $f^n(z) \to \xi \in \partial \mathcal{U}$



Define $p: [0,1] \to \mathcal{U}$ any path connecting z and $f(z)$, extend it to

$$p : [0, \infty) \to \mathcal{U}$$

by $p(t+1) = f(p(t))$.

Then apply Snail lemma

$\Longrightarrow$ ξ is either attracting (impossible) or parabolic

$\Longrightarrow$ $\mathcal{U}$ is parabolic basin.

Lemma Every immediate attracting basin is either simply connected or infinitely connected.

Pf.: $f: \mathcal{U} \to \mathcal{U}$ attracting immediate basin

If $\chi(\mathcal{U})$ is finite, since $\mathcal{U} \subseteq \mathbb{C}$,

$\chi(\mathcal{U}) \le 0$.

Riemann-Hurwitz

$f: X \to Y$, $d = \deg(f)$

$$\chi(X) = d\,\chi(Y) - \sum_{p \in X} (e_p - 1) \quad \leftarrow \text{ramification indices}$$

$f: \mathcal{U} \to \mathcal{U}$

$$\chi(\mathcal{U}) = d\,\chi(\mathcal{U}) - \Sigma(\ldots) \leq d\,\chi(\mathcal{U})$$

So $\chi(\mathcal{U}) = 0$ or $\boxed{\chi(\mathcal{U}) = 1}$

simply connected.

$\chi(\overline{\mathbb{D}}) = 1$ $\qquad \chi(\mathbb{C}) = 1$

$\chi(S^2) = 2$

Hyperbolic & Sub-Hyperbolic Rational Maps

Def.: A rational map f is HYPERBOLIC if there exists a metric on a neighborhood N of $J(f)$ such that $\exists$ constants $c > 0$, $k > 1$ so that

$$\| Df^n(v) \| \geq c\, k^n \|v\|$$

for every $\vec{v}$ tangent to some $x \in J(f)$.

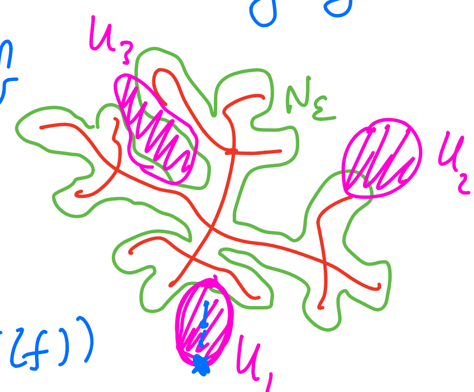
Thm A rational map of degree $d \geq 2$ is hyperbolic if and only if the forward orbit of every critical point converges to some attracting periodic point.

Prop Hyperbolic $\Longrightarrow$ every critical point converges to attracting cycles.

Rmk: there are no critical points in the Julia set.

Lemma Every critical point outside the Julia set converges to an attracting cycle.

Pf.: Show $\exists \varepsilon > 0$ so that if
$N_\varepsilon = \{ z : d(z, J(f)) < \varepsilon \}$



then for every $z \in N_\varepsilon$

$d(f(z), J(f)) \geq k\, d(z, J(f))$

and moreover $f(\hat{\mathbb{C}} \setminus N_\varepsilon) \cap N_\varepsilon = \emptyset$.

$L := \hat{\mathbb{C}} \setminus N_\varepsilon$, $f(L) \subset L$. Let

$U_1, \dots, U_p$ components of $\hat{\mathbb{C}} \setminus J(f)$ which intersect L

For each $j \leq p$ there is q_j s.t.

$f^{q_j}(U_j) = U_j$.

f cannot be of finite order,

$f^{q_j}|_{U_j}$ cannot be a rotation

because $f^{nq_j}(U_j) \subseteq L$ for n large, while if $f^{q_j}|_{U_j}$ is a rotation then $f^{nq_j}|_{U_j}(z)$

accumulates onto z, so if we pick $z \in U_j \setminus L$ we have a contradiction.

Also, U_j cannot be a parabolic basin because if so

$$f^n|_{U_j} \to \xi \in \partial U_j \cap J(f)$$

but $L \cap J(f) = \emptyset$

$$\left(\begin{array}{l}\text{All critical pts}\\ \text{attracted to attracting}\\ \text{cycles}\end{array}\right) \Longrightarrow \left(f \text{ Hyperbolic} \right)$$

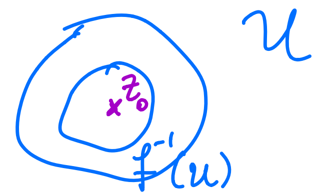
$P := \bigcup_{n \geq 1} f^n(c_j)$, $\{c_1, \dots, c_k\}$ = critical points

POSTCRITICAL SET

Hence $\overline{P} \cap J(f) = \emptyset$. Note $f(\overline{P}) \subset \overline{P}$

$\mathcal{U} := \hat{\mathbb{C}} \setminus \overline{P}$, hence $f^{-1}(\mathcal{U}) \subset \mathcal{U}$

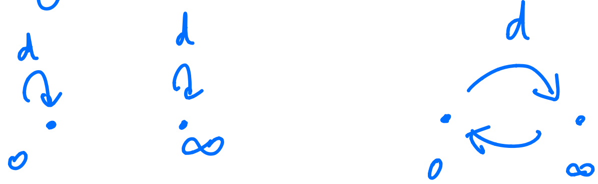
$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[d, "\pi"'] & \tilde{\mathcal{U}} \ni \tilde{z}_0 \arrow[l, "\varphi'"'] \arrow[d, "\pi"] \\ \mathcal{U} \arrow[r, "f"] & \mathcal{U} \ni z_0 \end{tikzcd}$$



Since $\overline{P} \cap J = \emptyset$, $\mathcal{U} \supset J$ hence it contains at least a repelling fixed point z_0 . If $\mathcal{U}$ is hyperbolic, We can lift it to $\tilde{z}_0$ an attracting fixed point for $\varphi : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$

So φ contracts the Poincaré metric on $\tilde{\mathcal{U}}$, hence f expands the Poincaré metric on $\mathcal{U}$.

Otherwise $\# \overline{P} = 1, 2$, but then $f(z) = z^{\pm d}$. In this case, $J = \{ |z| = 1 \}$ and you can take the euclidean metric



on a nbd of J.